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Recounting dyons in $\mathcal{N} = 4$ string theory

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Abstract: A recently discovered relation between 4D and 5D black holes is used to derive weighted BPS black hole degeneracies for 4D $\mathcal{N} = 4$ string theory from the well-known 5D degeneracies. They are found to be given by the Fourier coefficients of the unique weight 10 automorphic form of the modular group $\mathrm{Sp}(2, \mathbb{Z})$. This result agrees exactly with a conjecture made some years ago by Dijkgraaf, Verlinde and Verlinde.

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A general D0-D2-D4-D6 black hole in a 4D IIA string compactification has an M-theory lift to a 5D black hole configuration in a multi-Taub-NUT geometry. This observation was used in [1] to derive a simple relation between 5D and 4D BPS black hole degeneracies. For the case of $K3 \times T^2$ compactification, corresponding to $\mathcal{N} = 4$ string theory, the relevant 5D black holes were found in [2, 3] and the degeneracies are well known. In this paper we translate this into an exact expression for the 4D degeneracies, which turn out to be Fourier expansion coefficents of a well-studied weight 10 automorphic form Φ of the modular group of a genus 2 Riemman surface [4, 5].

Almost a decade ago an inspired conjecture was made [5] by Dijkgraaf, Verlinde and Verlinde for the 4D degeneracies of $\mathcal{N} = 4$ black holes, and this was shown to pass several consistency checks. We will see that our analysis precisely confirms their old conjecture.

$\mathcal{N} = 4$ string theory in four dimensions can be obtained from IIA compactification on $K3 \times T^2$. The duality group is conjectured to be

$$\mathrm{SL}(2;\mathbb{Z}) \times \mathrm{SO}(6, 22;\mathbb{Z}). \tag{1}$$

The first factor may be described as an electromagnetic S-duality which acts on electric charges $q_{e\Lambda}$ and magnetic charges q_m^Λ, $\Lambda = 0, \dots 27$ transforming in the 28 of the second factor. For the electric objects, we may take

$$q_e = (q_0; q_A; q_{23}; q_j), \tag{2}$$

where q_0 is D0-charge, $q_A, A = 1, \dots 22$ is $K3$-wrapped D2 charge, q_{23} is $K3$-wrapped D4 charge, and $q_i, i = 24, \dots 27$ are momentum and winding modes of $K3 \times S^1$-wrapped NS5 branes. The magnetic objects are 24 types of D-branes which wrap $T^2 \times$ (K3 cycle) and 4 types of F-string T^2 momentum/winding modes.

Now consider a black hole corresponding to a bound state of a single D6 brane with D0 charge q_0, $K3$-wrapped D2 charge q_A, and T^2-wrapped D2 charge q^{23}:

$$q_m = (1; q^A = 0; q^{23}; q^i = 0), \quad q_e = (q_0; q_A; q_{23} = 0; q_i = 0) \tag{3}$$

The duality invariant charge combinations are

$$\frac{1}{2}q_e^2 = \frac{1}{2}C^{\mathrm{AB}} q_A q_B, \quad \frac{1}{2}q_m^2 = q^{23}, \quad q_e \cdot q_m = q_0 \tag{4}$$

where C^{AB} is the intersection matrix on $H^2(K3;\mathbb{Z})$.

By lifting this to M-theory on Taub-NUT, it was argued in [1] that the BPS states of this system are the same as those of a 5D black hole in a $K3 \times T^2$ compactification, with T^2-wrapped M2 charge $\frac{1}{2}q_m^2$, $K3$-wrapped M2 charge q_A and angular momentum $J_L = q_0/2$. We now use one of the compactification circles to interpret the configuration as IIA on $K3 \times S^1$ with $\frac{1}{2}q_m^2$ F-strings winding S^1 and q_A D2-branes. T-dualizing the S^1 yields q_A D3-branes carrying momentum $\frac{1}{2}q_m^2$. This is then U-dual to a Q_1 D1 branes and Q_5 D5 branes on $K3 \times S^1$ with

$$N \equiv Q_1 Q_5 = \frac{1}{2}q_e^2 + 1 \tag{5}$$

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angular momentum[1]

$$J_L = \frac{1}{2} q_e \cdot q_m \tag{6}$$

and left-moving momentum along the S^1:

$$L_0 = \frac{1}{2} q_m^2. \tag{7}$$

Hence, with the above relations between parameters, according to [1] the 4D degeneracy of states with charges (3) and 5D degeneracies are related by

$$d_4(1; 0; q^{23}; 0|q_0; q_A; 0; 0) = (-1)^{q_0} d_5 \left(q^{23}, q_A; \frac{q_0}{2} \right). \tag{8}$$

The extra factor of $(-1)^{q_0}$ comes from the extra insertion of $(-1)^{2J_L}$ in the definition of the 5D index. Since the degeneracies are U-dual we may also write[2]

$$d_4(q_m^2, q_e^2, q_e \cdot q_m) = (-1)^{2J_L} d_5(L_0, N, J_L) = (-1)^{q_e \cdot q_m} d_5 \left(\frac{1}{2} q_m^2, \frac{1}{2} q_e^2 + 1, \frac{1}{2} q_e \cdot q_m \right). \tag{9}$$

Here and elsewhere in this paper by "degeneracies," in a slight abuse of language, we mean the number of bosons minus the number of fermions of a given charge, and the center-of-mass multiplet is factored out.

Of course these microscopic BPS degeneracies d_5 of the D1-D5 system are well known [2, 3]. Their main contribution comes from the coefficients in the Fourier expansion of the elliptic genus of $\mathrm{Hilb}^N(K3)$:

$$\chi_N(\rho, \nu) = \sum_{L_0, J_L} d_5'(L_0, N, J_L) e^{2\pi i (L_0 \rho + 2 J_L \nu)} \tag{10}$$

It is shown in [6] that the weighted sum of the elliptic genera has a product representation:

$$\sum_{N \geq 0} \chi_N(\rho, \nu) e^{2\pi i N \sigma} = \frac{1}{\Phi'(\rho, \sigma, \nu)} \tag{11}$$

where Φ' is given by

$$\Phi'(\rho, \sigma, \nu) = \prod_{k \geq 0, l > 0, m \in \mathbb{Z}} (1 - e^{2\pi i (k\rho + l\sigma + m\nu)})^{c(4kl - m^2)}, \tag{12}$$

with $c(4k - m^2) = d_5'(k, 1, m)$ the elliptic genus coefficients for a single $K3$ as given in [7].[3]

Equation (11) is the generating function for BPS states of CFTs on $\mathrm{Hilb}^N(K3)$ in the D5 worldvolume. However it does not quite give the degeneracies needed in (9), for two reasons. First, it leaves out the decoupled contribution from the elliptic genus of a single fivebrane. (By U-duality, we are free to view the system as a single fivebrane and N D1 branes.) Second, it misses the contribution from massless bulk modes obtained

[1] One should keep in mind that J_L is half the R-charge F_L [3], and is hence takes values in $\frac{1}{2}\mathbb{Z}$.

[2] Note that d_n denotes fixed-charge degeneracies and does not involve a sum over U-duality orbits.

[3] Note $c(-1) = 2$, $c(0) = 20$, and $c(n) = 0$ for $n \leq -2$.

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from the reduction of IIB on $K3\times$Taub-NUT. This reduction relies on the existence of a unique, normalizable, self-dual, harmonic 2-form $\omega_{\rm NUT}$ in Taub-NUT space [8]. These two contributions remain even when $N=0$ and there are no D1 branes at all.[4]

Now let us briefly sketch how these partition functions are calculated. (See [9] for a more careful derivation.) The single D5 has partition function (keep in mind that it carries both S^1 momentum and transverse angular momentum)

$$Z_{D5}(\rho,\nu)=(e^{\pi i\nu}-e^{-\pi i\nu})^{-2}\prod_{n\geq 1}(1-e^{2\pi i(n\rho+\nu)})^{-2}(1-e^{2\pi i(n\rho-\nu)})^{-2}(1-e^{2\pi i n\rho})^{4} \tag{13}$$

where the infinite product comes from the 4 transverse bosonic and fermionic collective coordinates of the D5 brane. Meanwhile, the left moving bulk modes consist of 24 scalars, which could be either seen by reducing the IIB massless fields using $\omega_{\rm NUT}$, or by the U-dual description in terms of a fundamental heterotic string. They do not carry J_L and contribute a partition function

$$Z_{\rm bulk}(\rho)=\eta(\rho)^{-24}=e^{-2\pi i\rho}\prod_{n\geq 1}(1-e^{2\pi i n\rho})^{-24}\,. \tag{14}$$

Combining these two contributions, we obtain the partition function

$$\begin{aligned} Z_0(\nu,\rho) &= Z_{D5}(\nu,\rho)Z_{\rm bulk}(\rho) \\ &= (e^{\pi i\nu}-e^{-\pi i\nu})^{-2}e^{-2\pi i\rho}\prod_{n\geq 1}(1-e^{2\pi i(n\rho+\nu)})^{-2}(1-e^{2\pi i(n\rho-\nu)})^{-2}(1-e^{2\pi i n\rho})^{-20} \end{aligned} \tag{15}$$

This shifts Φ' to

$$\frac{1}{\Phi'(\rho,\sigma,\nu)}\rightarrow\frac{Z_0(\nu,\rho)}{\Phi'(\rho,\sigma,\nu)}=\frac{e^{2\pi i\sigma}}{\Phi(\rho,\sigma,\nu)} \tag{16}$$

where $\Phi(\rho,\sigma,\nu)$ has a product representation

$$\Phi(\rho,\sigma,\nu)=e^{2\pi i(\rho+\sigma+\nu)}\prod_{(k,l,m)>0}\left(1-e^{2\pi i(k\rho+l\sigma+m\nu)}\right)^{c(4kl-m^2)} \tag{17}$$

where $(k,l,m)>0$ means that $k,l\geq 0$, $m\in\mathbb{Z}$ and in the case $k=l=0$, the product is only over $m<0$. $\Phi(\rho,\sigma,\nu)$ is the unique automorphic form of weight 10 of the modular group $\mathrm{Sp}(2,\mathbb{Z})$ and was studied in [4]. The 5D BPS degeneracies are then the Fourier coefficients in

$$\sum_{L_0,N,J_L}d_5(L_0,N,J_L)e^{2\pi i(L_0\rho+(N-1)\sigma+2J_L\nu)}=\frac{1}{\Phi(\rho,\sigma,\nu)}. \tag{18}$$

Inserting the 4D-5D relation (9), (18) agrees with the formula proposed in [5] for the microscopic degeneracy of BPS black holes of $\mathcal{N}=4$ string theory — up to an overall factor of $(-1)^{q_e\cdot q_m}$.[5]

[4]We thank the referee for pointing out an error regarding these contributions in the original version of the paper.

[5]Note that the formula of [5] was manifestly invariant under the duality group (1), and the extra factor of $(-1)^{q_e\cdot q_m}$ does not spoil this. Invariance of $(-1)^{q_e\cdot q_m}$ under $\mathrm{SO}(6,22;\mathbb{Z})$ follows by construction, and invariance under $\mathrm{SL}(2;\mathbb{Z})$ follows from the S-duality transformations of the charges.

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